Physics 436: Electromagnetism I
First Mid-Term Test

- Open book examination. You may use the course textbook book, notes, homework and handouts (including solutions).
- Choose 3 out of the 4 questions.
- Each question is worth 10 points. Point values of parts of a question are given in square parentheses.
- Time allowed: 1 hr. 15 min.
- If you do all 4 questions, best 3 will count.

These formulas may be useful (you don’t necessarily have to use all of them in this test):

- Circumference of a circle, radius $r = 2\pi r$
- Area of surface of sphere, radius $r = 4\pi r^2$
- Volume integral for spherically symmetric function $f(r)$ from radius $a$ to $b$
  \[
  \int_a^b dr \ r^2 f(r) = 4\pi \int_a^b dr \ r^2 f(r).
  \]
- \[
  \int dr \ r^n = \begin{cases} 
  \frac{r^{n+1}}{n+1} & \text{for } n \neq -1; \\
  \ln(r) & \text{for } n = -1.
  \end{cases}
  \]
- \[
  \int dr \ \frac{r}{\sqrt{r^2 + z^2}} = \sqrt{r^2 + z^2}.
  \]
Question 1

(a) A uniform ring of charge of radius \( r \) and charge \( Q \) is on the \( x-y \) plane with its center at the origin, as shown in figure (a) below. Show that the potential due to this ring along the \( z \)-axis is

\[
V_{\text{ring}}(z) = \frac{1}{4\pi\varepsilon_0} \frac{Q}{\sqrt{r^2 + z^2}}. \quad [4 \text{ points}]
\]

(b) Use the result in (a) to show that, for a uniform disk of charge of radius \( R \) and charge density \( \sigma \) as shown in figure (b) below, the potential along the \( z \)-axis is

\[
V_{\text{disk}}(z) = \frac{\sigma}{2\varepsilon_0} \left( \sqrt{R^2 + z^2} - R \right).
\]

Hint: Break the disk into rings of radius \( r \) and width \( dr \), figure out the amount of charge on this ring, use part (a) to find the potential for this ring, then integrate the potential due to these rings from \( r = 0 \) to \( r = R \). [6 points]

Question 2

The potential at the surface of a sphere of radius \( R \) is given by

\[
V(r = R, \theta) = \frac{3k}{2} \cos^2 \theta,
\]

where \( k \) is a constant. Show that the potential \textit{inside} sphere is

\[
V(r, \theta) = k \frac{r^2}{R^2} P_2(\cos \theta) + \frac{k}{2},
\]

where \( P_2 \) is the second-order Legendre polynomial. Assume there is no charge inside the sphere. [Hint: No need to use Fourier’s trick here – just use the “eyeball” technique.]
Question 3

A sphere of radius $R$ carries a charge density $\rho(r) = kr$ (where $k$ is a constant) for $r < R$. On the surface of the sphere, there is a uniform charge that is equal to in magnitude but opposite in sign from the charge inside the sphere (so that whole sphere, including the interior and the surface, is electrically neutral).

(a) Show that the charge enclosed inside a sphere of radius $r < R$ (centered at the origin) is $Q(r) = \pi kr^4$. [3 points]

(b) Use part (a) and Gauss’ law (or another technique, if you wish) to show that the magnitude of the electric field as a function of $r$ is

$$E(r) = \begin{cases} \frac{kr^2}{4\varepsilon_0} & \text{for } r < R; \\ 0 & \text{for } r > R. \end{cases}$$ [4 points]

(c) How much energy is stored in this charge configuration? [3 points]

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Question 4

A uniformly charged sphere of radius $R$ with a total charge $Q > 0$ is brought near an infinite grounded (i.e., $V = 0$) conducting plane on the $x$-$y$ plane, so that the center of the sphere is on the $z$-axis and at $z = d$ (where $d > R$). See figure below. What is the electric field (magnitude and direction) at the center of the sphere [i.e., at $(0,0,d)$]? [Hint: Use the method of images. Then, use superposition of electric fields due to the uniformly charged sphere and its image.]