Physics 431/531: Mechanics 1
Mid-Term Test 2 Solutions

Question 1

(a) For critical damping, the general solution for an un-driven harmonic oscillator is [see (5.44) in the textbook] \( x(t) = C_1 \exp(-\beta t) + C_2 t \exp(-\beta t). \) Substituting \( t = 0 \) into this gives \( x(0) = C_1. \) Since \( x(0) = 0, \) this implies \( C_1 = 0, \) and therefore \( x = C_2 t e^{-\beta t}. \) Now take the derivative with respect to time of this, giving \( \dot{x} = C_2 (1 - \beta t) e^{-\beta t}. \) Substitute \( t = 0 \) into this, and setting this equal to \( v_0 \) gives \( C_2 = v_0. \) Therefore \( x(t) = v_0 t e^{-\beta t}. \)

(b) The steady-state solution is given by Eq. (5.66) in the textbook, except that instead of cos, in our case we have sin, and \( \omega = \beta; \) i.e.
\[
 x(t) = A \sin(\beta t - \delta). 
\]
\( A \) is given by Eq. (5.64), with \( \omega = \omega_0 = \beta; \) \( A = f_0/(2\beta^2). \) The phase \( \delta \) is given by Eq. (5.65): \( \delta = \arctan(\pm \infty) = \pi/2 \) (see figure 5.19). Putting these into Eq. (1) gives
\[
 x(t) = \frac{f_0}{2\beta^2} \sin(\beta t - \pi/2) = -\frac{f_0}{2\beta^2} \cos(\beta t). 
\]

Question 2

(a) A change in \( z \) of \( dz \) gives a displacement \( dz \hat{z}, \) and a change in \( \phi \) of \( d\phi \) gives a displacement \( R d\phi \hat{\phi}. \) Therefore, a simultaneous change in \( z \) and \( \phi \) of \( dz \) and \( d\phi \) gives a displacement of magnitude \( ds = \sqrt{(dz)^2 + R^2(d\phi)^2}, \) by Pythagoras' theorem. The length of a path on a cylinder is given by \( L = \int ds = \int \sqrt{(dz)^2 + R^2(d\phi)^2}. \) Factoring \( dz \) out of the integrand gives
\[
 L = \int \sqrt{1 + R^2 \left( \frac{d\phi}{dz} \right)^2} \, dz = \int \sqrt{1 + R^2 \phi'^2} \, dz, 
\]
where \( \phi' = d\phi/dz. \)

(b) Applying the Euler-Lagrange formula to the integrand gives
\[
 \frac{d}{dt} \left( \frac{d}{d\phi'} \sqrt{1 + R^2 \phi'^2} \right) = 0 \implies \frac{R^2 \phi'}{\sqrt{1 + R^2 \phi'^2}} = \text{constant}. 
\]
Solving for \( \phi' \) gives \( \phi'(z) = a \) (a constant). Integrating this with respect to \( z \) gives \( \phi(z) = az + b. \)
Question 3

(a) The Lagrangian \( L = T - U \), where \( T \) must be evaluated in an inertial frame. In this problem \( U = 0 \) (since the bead is moving in a horizontal plane). The radial component of the velocity of the bead is \( \dot{x} \), and the tangential component is \( x \Omega \), so \( L = m \left( \dot{x}^2 + x^2 \Omega^2 \right) \). Applying the Euler-Lagrange equation to this gives

\[
\frac{\partial L}{\partial x} = \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \Rightarrow (\text{cancelling } m) \quad x \Omega^2 = \frac{d}{dt} \dot{x} = \ddot{x}.
\]

(b) Guess the solution of the form \( e^{\alpha t} \) and substitute into \( \ddot{x} = \Omega^2 x \) gives \( \alpha^2 e^{\alpha t} = \Omega^2 e^{\alpha t} \), which is satisfied when \( \alpha = \pm \Omega \). The general solution is a linear superposition of the two solutions \( e^{\Omega t} \) and \( e^{-\Omega t} \); i.e., \( A e^{\Omega t} + B e^{-\Omega t} \).

(c) Substituting \( t = 0 \) into the general solution and setting it equal to \( x_0 \) gives \( A + B = x_0 \). Taking the time derivative of the general solution and setting that equal to 0 gives \( A - B = 0 \) \( \Rightarrow A = B \). Therefore \( A = B = \frac{x_0}{2} \), and hence for the given initial conditions,

\[
x(t) = \frac{x_0}{2} (e^{\Omega t} + e^{-\Omega t}) = x_0 \cosh(\Omega t).
\]

Question 4

(a) It is given that \( r_{\text{max}}/r_{\text{min}} = 2 \). Taking the ratio of \( r_{\text{min}} = c/(1 + \epsilon) \) and \( r_{\text{max}} = c/(1 - \epsilon) \) gives \( (1 + \epsilon)/(1 - \epsilon) = r_{\text{max}}/r_{\text{min}} = 2 \). Solving for \( \epsilon \) gives \( \epsilon = \frac{1}{3} \).

(b) Use Eq. (8.48) in the textbook, with \( \gamma = GM_s m_a \) (where \( m_a \) is the mass of the asteroid), \( \mu = m_a \) (since \( m_a \ll M_s \)) and \( \ell = m_a r_0 v_0 \), where \( v_0 \) is the speed at the perihelion. This gives \( c = \frac{r_0^2 v_0^2}{GM_s} \). But from Eq. (8.50), \( r_{\text{min}} = r_0 = c/(1 + \epsilon) \Rightarrow c = (1 + \epsilon) r_0 = \frac{4}{3} r_0 \). Therefore

\[
\frac{4}{3} r_0 = \frac{r_0^2 v_0^2}{GM_s} \Rightarrow v_0 = 2 \sqrt{\frac{GM_s}{3 r_0}}.
\]

Conservation of angular momentum gives \( 2 r_0 v_{\text{aphelion}} = r_0 v_0 \Rightarrow v_{\text{aphelion}} = v_0/2 = \sqrt{\frac{GM_s}{3 r_0}} \).

(c) If \( v_0 = 2 \sqrt{GM_s/r_0} \), the total energy of the asteroid would be

\[
\frac{1}{2} m_a v_0^2 - \frac{GM_s m_a}{r_0} = 2 \frac{GM_s m_a}{r_0} - \frac{GM_s m_a}{r_0} = + \frac{GM_s m_a}{r_0} > 0.
\]

Since the total energy is positive, the motion of the asteroid is \textbf{unbounded}. 