Mechanics 1
Mid-Term Test 1 Solutions

Question 1

From Newton’s second law

\[ m \frac{dv}{dt} = -F_0 e^{\alpha v}. \]

Separate variables:

\[ e^{-\alpha v} dv = -\frac{F_0}{m} dt. \]

Integrate both sides:

\[ \int_{v_0}^{v} e^{-\alpha v'} dv' = -\frac{F_0}{m} \int_{0}^{t} dt' \Rightarrow -\frac{1}{\alpha} [e^{-\alpha v} - e^{-\alpha v_0}] = -\frac{F_0 t}{m}. \]

When the object stops, \( v = 0 \). Substituting this into the above equation gives

\[ -\frac{1}{\alpha} [1 - e^{-\alpha v_0}] = -\frac{F_0 t_{\text{stop}}}{m}. \]

Solving for \( t_{\text{stop}} \) gives the desired result,

\[ t_{\text{stop}} = \frac{m}{F_0 \alpha} (1 - e^{-\alpha v_0}). \]

It is always worthwhile to see that a result that you obtain makes sense in certain limits. Here, if you let \( \alpha \to 0 \), then the force will be equal to \( -F_0 \), and so the acceleration is \( -F_0/m \) and the time taken for the object to stop is \( t_{\text{stop}} = -v_0/a = mt_0/F_0 \). Taking the \( \alpha \to 0 \) limit of the expression in the box for \( t_{\text{stop}} \), using L’Hospital’s rule (because there is an \( \alpha \) in the denominator) also gives \( t_{\text{stop}} = \frac{m}{F_0} (\partial(1 - e^{-\alpha v_0})/\partial\alpha)_{\alpha=0} = mv_0/F_0 \).

Question 2

(a) By Newton’s second law, external forces contribute to the \( m \ddot{v} \), so the modification of the equation would be

\[ m \ddot{v} = -mv_{\text{ex}} + F_{\text{ext}}. \]
Use the above equation. Here, the velocity is constant, so \( \dot{v} = 0 \). \( \dot{m} \) is the rate at which mass is added to the vertical section of the rope. In a time \( dt \), a segment of length \( v \, dt \) is added to the vertical section, and hence the mass added to the vertical section is \( dm = \lambda v \, dt \). Dividing both sides by \( dt \) gives \( \dot{m} = \lambda v \). \( v_{ex} \) is defined in the textbook such that the mass element \( dm \) is such that its velocity relative to the ground is \( v - v_{ex} \). Since the mass element \( dm \) is stationary relative to the ground before it joins the moving vertical section, \( v_{ex} = v \). Therefore \( -\dot{m}v_{ex} = -\lambda v^2 \).

The external force is made up of the force \( F \) pulling the rope up, and the weight force of gravity \( F_{\text{ext}} = F - mg = F - \lambda zg \). Putting all of these into the above equation gives

\[
0 = -\lambda v^2 + F - \lambda gz \quad \Rightarrow \quad F = \lambda (gz + v^2).
\]

(Incidentally, I think that a clearer and more general derivation of this “rocket equation” can be given. Let the velocity of the mass \( M \) at time \( t \) be \( v \), and that of a small mass element \( dm \) be \( u_{\text{initial}} \). A small time \( dt \) later, the velocity of mass \( M \) is \( v + dv \) and that of the small mass element \( dm \) is \( u_{\text{final}} \). The change in momentum is

\[
P(t + dt) - P(t) = dP = M(v + dv) + dm u_{\text{final}} - Mv - dm u_{\text{initial}}
= Md\dot{v} + dm \Delta u,
\]

where \( \Delta u = u_{\text{final}} - u_{\text{initial}} \) is the change in velocity of the small mass element \( dm \). Dividing both sides of the above equation by \( dt \), and using Newton’s second law, \( F_{\text{ext}} = dP/dt \), gives

\[
F_{\text{ext}} = M\ddot{v} + \dot{m}\Delta u.
\]

This form works equally well for mass being expelled or accumulated from the main body. For example, in the problem with the rope being pulled upwards, \( \dot{m} = \lambda v \), and \( \Delta u = v - 0 = v \), so \( \dot{m}\Delta u = \lambda v^2 \).

**Question 3**

(a) To show that the force is conservative, take the curl:

\[
\nabla \times F = \begin{vmatrix}
\hat{x} & \hat{y} & \hat{z} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
Axy^2 & Ax^2y & Bz
\end{vmatrix} = 0\hat{x} + 0\hat{y} + \left( \frac{\partial Axy^2}{\partial x} - \frac{\partial Ax^2y}{\partial y} \right)\hat{z} = 0.
\]
(b) Since this is conservative, one can associate a potential energy $U$ with it, given by

$$U(x, y, z) = -\int_{\text{ref}}^{(x, y, z)} F(r') \cdot dr',$$

along any path. Choose the origin to be the reference point, and choose the path $(0, 0, 0) \rightarrow (x, 0, 0) \rightarrow (x, y, 0) \rightarrow (x, y, z)$. The line integral from $(0, 0, 0) \rightarrow (x, 0, 0)$ is

$$\int_{x' = 0}^{x'} F_x(x', 0, 0) \, dx' = \int_{x' = 0}^{x} 0 \, dx' = 0.$$

since $F_x = Axy^2$ and $y = 0$ along this path. The line integral from $(x, 0, 0) \rightarrow (x, y, 0)$ is

$$\int_{y' = 0}^{y} dy' \, F_y(x, y', 0) = Ax^2 \int_{0}^{y} y' \, dy' = \frac{1}{2} Ax^2 y^2.$$

Finally the line integral from $(x, y, 0) \rightarrow (x, y, z)$ is

$$\int_{z' = 0}^{z} dz' \, F_z(x, y, z') = B \int_{0}^{z} z' \, dz' = \frac{1}{2} Bz^2.$$

So $U = -\frac{1}{2} Ax^2 y^2 - \frac{1}{2} Bz^2$. (As a check, notice that $-\nabla U$ reproduces the force field.) To find the particle’s speed at $(x_1, y_1, z_1)$, use conservation of energy.

$$E_{\text{initial}} = \frac{1}{2} mv_0^2 = E_{\text{final}} = \frac{1}{2} mv_1^2 - \frac{1}{2} Ax_1^2 y_1^2 - \frac{1}{2} Bz_1^2$$

$$\Rightarrow v_1 = \sqrt{v_0^2 + \frac{1}{m} (Ax_1^2 y_1^2 + Bz_1^2)}.$$

**Question 4**

The easiest way is to use conservation of energy. The kinetic energy of the system is

$$T = \frac{1}{2} mv^2 + \frac{1}{2} I \omega^2.$$

Here, $\omega = v/R$ since the pulley is attached by a string to the mass, and

$$v = \dot{x}, \text{ so } T = \frac{1}{2} \left( m + \frac{I}{R^2} \right) \dot{x}^2.$$
The potential energy is equal to $mgy$, where $y$ is the vertical distance from a reference point (where upwards is positive). From trigonometry, $y = -x \sin \theta$; therefore $U = -mgx \sin \theta$ and

$$E = \frac{1}{2} \left( m + \frac{I}{R^2} \right) \dot{x}^2 - mgx \sin \theta.$$  

Since $E$ is conserved, $dE/dt$ is zero. Taking the time derivative of the above equation and using the chain rule gives

$$0 = \left( m + \frac{I}{R^2} \right) \ddot{x} \dot{x} - mg \sin \theta \dot{x}.$$  

Factoring out and cancelling $\dot{x}$ and solving for $\ddot{x}$ gives

$$\ddot{x} = \frac{g \sin \theta}{1 + \frac{I}{mR^2}}.$$